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Honeycombs in the Hosoya Triangle

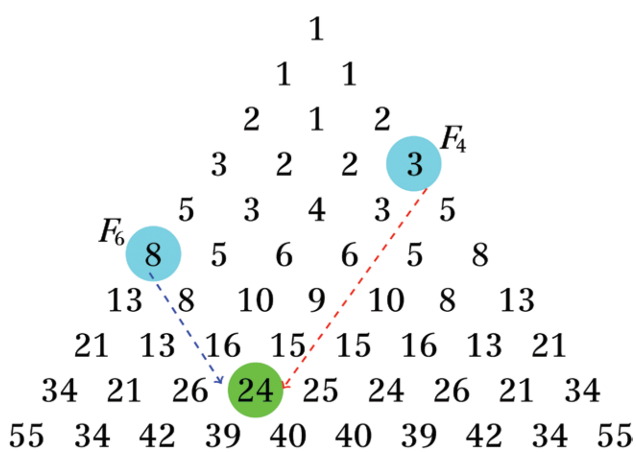
Matthew Blair, Rigoberto Flórez, and Antara Mukherjee

Pascal's triangle (perhaps more aptly named Halayudha's triangle or simply the arithmetic triangle) generally holds the title as the most famous number triangle. You can even find the Fibonacci numbers—defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$—in the triangle, and this sequence provides a treasure trove of patterns and identities. But there is another, less well-known number triangle that contains the Fibonacci numbers as well. We have found that experimenting with geometric patterns in this Fibonacci triangle provides the opportunity to discover old and new Fibonacci identities.

In 1976, Haruo Hosoya, a chemist with interest in discrete mathematics, introduced what is now called the *Hosoya triangle* (formerly the Fibonacci triangle) pictured in figure 1 (Hosoya, "Fibonacci Triangle," *Fib. Quart.* **14(3)** [1976]). The kth entry in row r of this triangle, denoted by $H(r,k)$, is defined to be $F_k \cdot F_{r-k+1}$ so that each entry of the Hosoya triangle is the product of two Fibonacci numbers. For example, the entry highlighted in green in figure 1 is $H(9,4) = F_4 \cdot F_6 = 24$.

Alternatively, each entry can be written as the sum of the two previous entries in the *slash* or the *backslash* diagonal: the slash diagonal is the diagonal in the southwest direction while the backslash diagonal is the diagonal in the southeast direction, as illustrated in figure 1. More precisely, we have $H(r,k) = H(r-1,k-1) + H(r-2,k-2)$ and $H(r,k) = H(r-1,k) + H(r-2,k)$. For instance, the entry 24 highlighted in green in figure 1 is $24 = 15 + 9$ (along the slash diagonal) or $24 = 16 + 8$ (along the backslash diagonal).

Figure 1. The Hosoya triangle. One slash diagonal is pictured in dashed red and a backslash diagonal is pictured in dashed blue.



We initiate the exploration of certain geometric patterns embedded in the Hosoya triangle. These techniques can engage students in undergraduate research to help discover, or rediscover, identities. In fact, these experiments were conducted by the first author while he was working on his senior research project.

Honeycomb Movements

Number triangles admit a tiling by hexagonal tiles as seen in figure 2. We call this tiling the *honeycomb pattern*. Figure 3 shows certain paths, inspired by the work of Leonard Carlitz, that bees might take through the honeycomb pattern. Originally, Carlitz worked on an infinite honeycomb with only two adjacent rows of cells to find the number of paths that a bee could take

Figure 2. Tiling the Hosoya triangle with hexagons creates the honeycomb pattern.

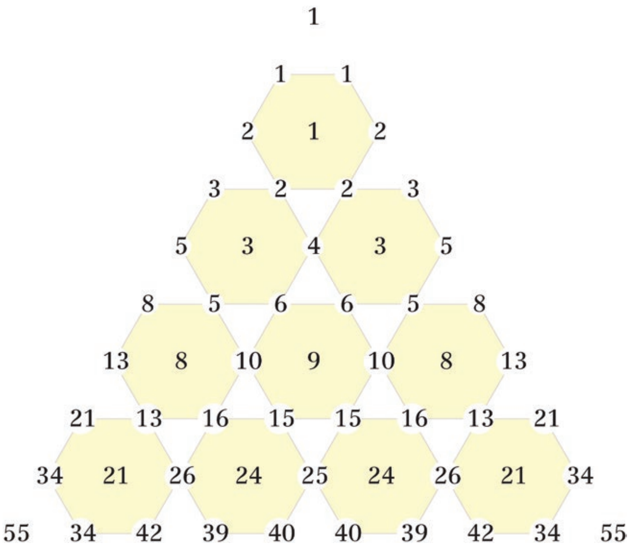
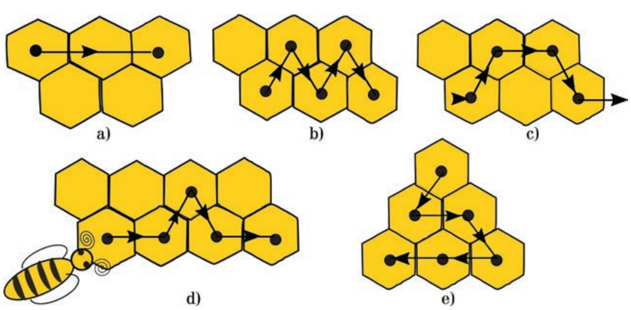


Figure 3. Carlitz patterns on honeycombs.



to travel from one cell to another, given that it can only move left to right. Carlitz found that the number of paths used to travel from cell 1 to cell n is F_{n+1} (Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, [2001]).

We consider a different perspective by embedding the Carlitz paths from figure 3 into the Hosoya triangle.

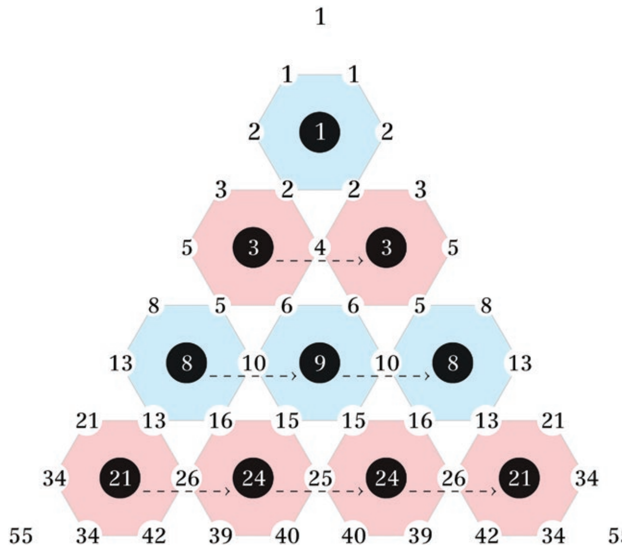
The pattern in figure 3a embeds into odd-numbered rows of the Hosoya triangle (as in figure 4). The third pictured row shows $F_2F_6 + F_4F_4 + F_6F_2 = 8 + 9 + 8 = 25$. In general, this pattern produces the sum

$$\sum_{k=1}^{(n-1)/2} F_{2k}F_{n-2k+1},$$

when n is odd. Computing the first few values of this sequence yields 1, 6, 25, 90, 300, …. We discovered (and can prove by induction) that this sequence has the following closed formula:

$$\sum_{k=1}^{(n-1)/2} F_{2k}F_{n-2k+1} = \frac{(4n-8)F_{n-3} + (7n-11)F_{n-2}}{10}.$$

Figure 4. The pattern from figure 3a results in summing every other entry in the odd rows.



This sequence appears in the Online Encyclopedia of Integer Sequences (OEIS); the entry contains other closed formulas (*oeis.org/A001871*).

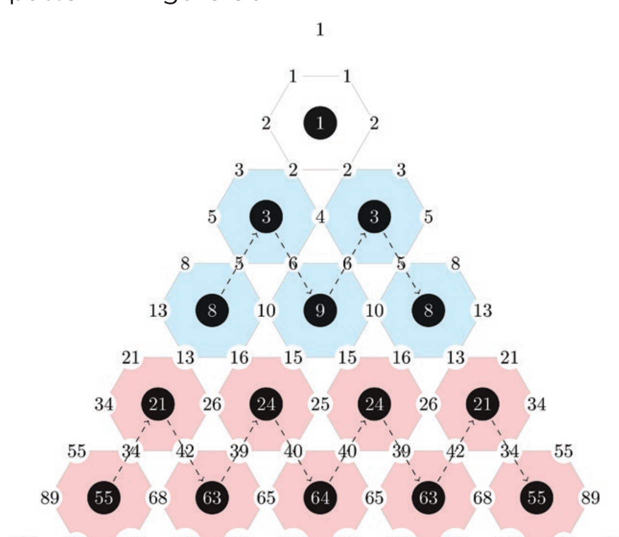
Next, by embedding the pattern from figure 3b into the triangle (figure 5), we obtain sums like

$$F_2F_{10} + F_2F_8 + F_4F_8 + F_4F_6 + F_6F_6$$
$$+F_6F_4 + F_8F_4 + F_8F_2 + F_{10}F_2 = 390.$$

In general, this pattern leads to the formula

$$\sum_{k=1}^{(n-1)/2} F_{2k}F_{n-2k+1} + \sum_{k=1}^{(n+1)/2} F_{2k}F_{n-2k+3} = \sum_{k=0}^{n} kF_{2k} = \frac{(n+1)F_{n+2} - 2F_{n+1}}{2},$$

Figure 5. Two sums (not successive) of the pattern in figure 3b.



for any odd positive integer $n > 1$ (which appears in *oeis.org/A197649*).

We can embed the pattern in figure 3c with one plateau in row $r = 7$ as in figure 6. When we sum the entries in this pattern, we obtain $F_2F_6 + F_2F_4 + F_4F_2 + F_6F_2 = 2F_2(F_4 + F_6)$. The next row in which we can embed the pattern is row $r = 13$, allowing two plateaus (as in figure 6). The resulting sum is given by

$$2(F_2F_{12} + F_2F_{10} + F_4F_8 + F_8F_6)$$
$$= 2(F_2(F_{12} + F_{10}) + F_8(F_4 + F_6)).$$

We can include three plateaus in row $r = 19$, yielding the sum

$$2(F_2F_{18} + F_2F_{16} + F_4F_{14} + F_6F_{14} + F_8F_{12} + F_8F_{10} +)$$
$$= 2(F_2(F_{16} + F_{18}) + F_{14}(F_4 + F_6) + F_8(F_{10} + F_{12})).$$

We note that $F_{k-1} + F_{k+1} = L_k$, where (L_n) denotes the sequence of Lucas numbers, defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$. Using the Lucas numbers, the sums given above for rows $r = 7, r = 13$, and $r = 19$ equal $2F_2L_5$, $2(F_2L_{11} + F_8L_5)$, and $2(F_2L_{17} + F_8L_{11} + F_{14}L_5)$ respectively.

In general for $n \geq 1$, we can embed the pattern from figure 3c in row $r = 6n + 1$ using n plateaus. The sum of the entries in the pattern will be

$$2\sum_{i=0}^{n-1} F_{6i+2}L_{m-6i},$$

where $m = 6n - 1$.

Finally, the triangular pattern in figure 3e traverses every hexagon in the honeycomb tiling as shown in figure 7. The entries from these hexagons produce the integer sequence 1, 3, 3, 8, 9, 8, 21, 24, ... According to the OEIS (A141678), this represents a triangular array in which each entry is described in terms of the bisection of the Fibonacci sequence. If we let entry k in row n of this new triangle be denoted by $T(n,k)$, then our interpretation shows that

$$T(n,k) = F_{2n-2k+2} \cdot F_{2k+2}.$$

Keeping track of the partial sums along the path in figure 7 gives 1, 4, 7, 15, 24, 32, ..., which is a sequence not in the OEIS! Can you find a closed formula?

Figure 6. Embedding pattern from figure 3c in rows 7 and 13 from the Hosoya triangle.

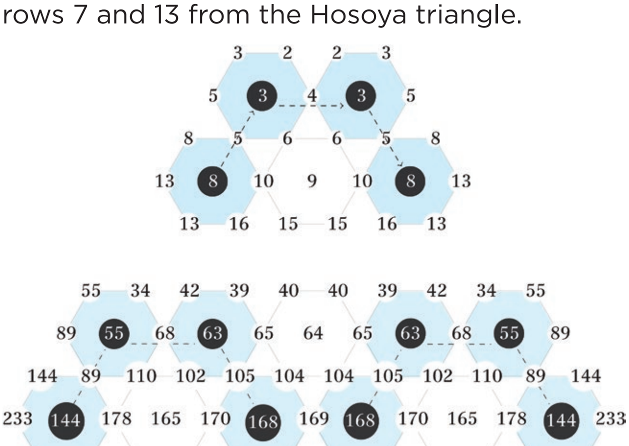
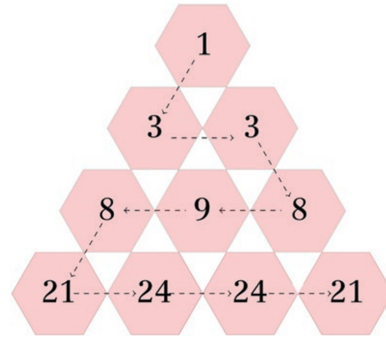


Figure 7. Extending the triangle path from figure 3e in the Hosoya triangle.



Further Explorations

The use of honeycombs to search for patterns in the Hosoya triangle arose from the second and third authors' interest in the generalization of the *star of David property*, a well-known result for Pascal's triangle, to the Hosoya triangle.

In the case of Pascal's triangle, if a star of David is embedded in a hexagon of numbers, as illustrated on the left in figure 8, then the greatest common divisor (denoted by gcd) of the alternating points of the hexagon of the honeycomb are equal, that is, $\gcd(a,c,e) = \gcd(b,d,f)$. Moreover, the product of alternating points of a hexagon of the honeycomb are equal, so $a \cdot c \cdot e = b \cdot d \cdot f$.

These properties have analogous facts in the Hosoya triangle. The product of alternating points of the hexagon in the honeycomb are also equal in this case, that is, $a \cdot c \cdot e = b \cdot d \cdot f$ in figure 8. The greatest common divisor of alternating points of the hexagon in the honeycomb are equal and always 1: $\gcd(a,c,e) = \gcd(b,d,f) = 1$. And finally, the product of the two greatest common divisors of both opposite diagonal vertex numbers in the

Figure 8. The general star of David property and one star of David from the Hosoya triangle.

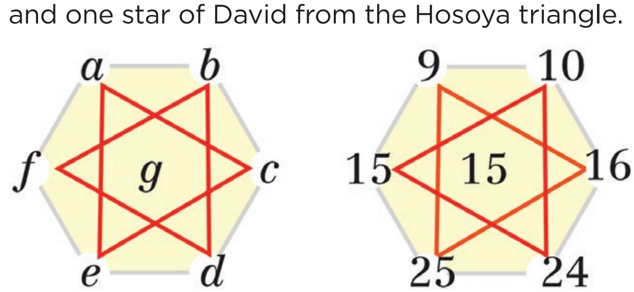
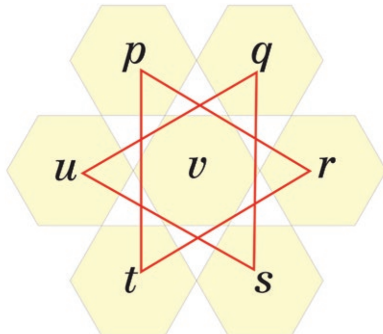


Figure 9. A larger star of David pattern.



hexagon is equal to the value within the hexagon: $\gcd(a,d)\cdot\gcd(b,e)=g$.

In the first hexagon from row $r=2$ of the honeycomb pattern in figure 2, we can check that $3\cdot5\cdot4=60=2\cdot6\cdot5$; $\gcd(3,5,4)=1=\gcd(2,6,5)$; and $\gcd(3,6)\cdot\gcd(2,5)=3$. These properties actually hold in any hexagon in the Hosoya triangle. For example, you can check these relations hold for hexagon on the right of figure 8, centered at entry $H(8,5)$ in the triangle.

We invite the reader to investigate properties using a larger star of David (like the one in figure 9) in the Hosoya triangle.

There are numerous other number triangles to investigate, such as Lucas's, Josef's, and Liebniz's harmonic triangle. You can explore these triangles (or even Pascal's triangle) using or expanding on the techniques discussed here. This approach of experimentation with triangular arrays can be quite effective when it comes to engaging in research as an undergraduate or beyond (Flórez and Mukherjee, "Solving open problems with students as a first research experience," *Teach. Math. Its Appl* [2017]).

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